ISYE 7201: Production & Service Systems
Spring 2020
Instructor: Spyros Reveliotis
2nd Midterm Exam (Take Home)
Release Date: March 4, 2020
Due Date: March 14, 2020

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

You can send me your responses as a pdf file attached to an email. This pdf file can be a scan of a hand-written document, but, please, write your answers very clearly and thoroughly. Also, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (20 points): In class we showed that a counting process $\{N(t), t \geq 0\}$ where the inter-event times are independent, exponentially distributed random variables with a common rate λ, is Poisson with the same rate. However, during the proof of this result, I skipped the part that would establish that the considered process $N(t)$ has independent increments. Provide the missing argument.

Problem 2 (20 points): Consider an elevator that starts in the basement and travels upwards. Let N_i denote the number of people that get in the elevator at floor i. Assume that N_i are independent and that N_i is Poisson distributed with mean λ_i. Each person entering at floor i will, independent of everything else, get off at floor j with probability p_{ij}. Furthermore, $\sum_{j>i} p_{ij} = 1.0$. Finally, let O_j denote the number of people getting off the elevator at floor j.

i. **(5 pts)** Compute $E[O_j]$.

ii. **(10 pts)** What is the distribution of O_j?

iii. **(5 pts)** What is the joint distribution of O_j and O_k?

Please, provide complete justifications for your answers.

Problem 3 (20 points): Consider the motion of three indistinguishable balls on a linear array of positions indexed by the positive integers, such that one or more balls can occupy the same position. Suppose that at time $t = 0$ there is one ball at position one, one ball at position 2 and one ball at position three. Given the positions of the balls at some integer time t, their positions at time $t + 1$ are determined as follows: One of the balls in the leftmost occupied position is picked up, and one of the other two balls is selected at random (but not moved) with each choice having probability one half. The ball that was picked up is then placed one position to the right of the selected ball.

i. **(5 pts)** Define a three-state Markov process that tracks the relative positioning of the balls at each discrete time $t \in Z_0^+$. Describe the meaning of each state, and give the one-step transition probability matrix for this process. (*Hint:* Exploit the fact that the balls are indistinguishable, and don't include the actual positions occupied by the balls in your definition of the process state.)

ii. **(5 pts)** Find the equilibrium distribution of the process defined in part (i).

iii. **(5 pts)** As time progresses, the entire set of balls moves to the right, and the average speed for this motion has a limiting value with probability one. Find this limiting value (*Hint:* Consider the ball motions in each state of the discrete-time Markov chain that you defined in the previous parts of this problem, and induce a notion of "speed" from these motions.)

iv. **(5 pts)** Consider the following continuous-time version of the above problem: Given the current state at time t, a move as described in the opening part of this problem, happens in the interval $[t, t+h]$ with probability $h + o(h)$. Provide the infinitesimal generator matrix Q for the corresponding CTMC, compute the equilibrium distribution for this process, and identify the long-term average speed of the ball drifting in this new regime.

Problem 4 (20 points) In each of the following four cases, compute

$$\lim_{t\to\infty} P(X_t = 2 | X_0 = 1)$$

for the Markov chain $(X_t)_{t\geq 0}$ with the given infinitesimal generator matrix on $\{1, 2, 3, 4\}$:

a. (5 pts)

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

b. (5 pts)

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

c. (5 pts)

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

d. (5 pts)

$$\begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the above representation, states that have their rows in the infinitesimal generator matrix equal to the **0** vector, are absorbing states for the corresponding process.

Problem 5 (20 points): When I introduced the concept of the CTMC, I discussed the modeling of an M/M/1 queueing station as an example of such a process. Now consider another single-server queueing station where customers arrive in a Poisson stream of rate λ. Each customer has a service requirement distributed according to an Erlang(2,μ) distribution. Service times are independent from each other, and of the arrival process. Also, customers joining the queue of this station are processed First-Come-First-Serve, and the server operates in a non-failing and non-idling mode.

i. **(10 pts)** Model the operation of this queueing station as a CTMC, explaining clearly the meaning of each state in your model, and detailing all the parameters that define the transitional dynamics of this CTMC.

ii. **(10 pts)** Also, show that this CTMC will have a limiting distribution if and only if $\lambda/\mu < 1/2$.